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Juggling Mathematical Understanding

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Abstract

This paper presents a theoretical model for the teaching for understanding of school mathematics. After describing two categories of understanding, it develops a continuum between rote and insight. In the process of describing the model, it articulates the assumptions underpinning the model and presents a process whereby a teacher can move the teaching strategies towards the development of insight within the students. It will argue that the development of insight should be the goal of all school mathematics classrooms. And that in order to achieve this goal the classroom teacher must become an expert juggler by simultaneously applying teaching strategies that develop student proficiency with skills, positive attitudes towards mathematics and deep connected conceptual knowledge.

Keywords: school mathematics, teaching for understanding, instrumental understanding, rote memorization, relational understanding.

Introduction

The secondary school mathematics curriculum includes many facts, skills, procedures and concepts. Mathematics teachers are expected to teach the curriculum while inculcating positive attitudes towards mathematics and by engaging and motivating their students to work mathematically. This article will seek to examine more closely the research into how teachers manage to juggle all these variables in order to assist their students to make sense of what they are learning. It is argued that teaching for understanding should be the predominant strategy in school mathematics classrooms.

There are many examples available of strategies that are regarded as ineffective or even harmful to the development of mathematical understanding. In earlier papers (White, 2011, 2013) the negative effects of behaviourism and skills based only strategies for teaching mathematics have been discussed. The mastery learning approach was given as an example of an unsatisfactory approach because the focus was upon reproduction of procedures and the issues of student thinking and understanding were never considered. It is necessary to indicate that there are modern versions of a mastery learning approach available on the web that mimic a behaviourist approach and others that have been influenced by other contemporary learning theories and strive to improve student thinking and mathematical understanding. So the challenge for both teachers and parents is how to differentiate between strategies or approaches.

In order to provide some direction in meeting this challenge, this article discusses the construction of a scale of teaching for understanding that will attempt to provide an instrument for categorising teaching strategies or approaches, using current research as the building material. Before the scale can be constructed it is necessary to briefly examine mathematics understanding.

Instrumental And Relational Understanding

The foundation of this scale of understanding relies upon Skemp (1976, 1977, 1979, 1986, 1989, 1992) and his classification of mathematics understanding. Specifically, it is necessary to elaborate on the two categories of instrumental and relational understanding. Instrumental understanding is described as 'rules without reasons' or ‘knowing how’ and for many students and sometimes their teachers the possession of such rules and the ability to use them with textbook and examination questions was a regarded as a demonstration of their 'understanding'. Why a rule worked was not considered and there was little effort to help the students to construct meaning. This instrumental approach, according to Skemp (1976, 1986), is initially easier to understand with more immediate and apparent rewards, and students who become used to this approach resist alternative teaching strategies. A predominant feature of this approach to teaching is drill and practice.

In contrast to instrumental understanding, relational understanding is concerned with meaning and developing connected understanding or knowledge. Relational understanding is 'knowing both what to do and why.' Skemp (1976, 1977) discusses the developing of schemas as evidence of the construction of relational understanding. While this approach also uses drill and practice and memorization, they are used in the service of understanding and supporting thinking.

The Use Of Card Tricks To Demonstrate Essential Features Of Both Types

A popular ‘mindreading’ card trick is summarised in figure 1 below. On the left are listed the mathemagician’s instructions and on the right are listed the algebraic steps that will be used to assist the construction of understanding.

Pick a number from 1 - 9	Yours is x, mine y
Add 1	$X + 1$
Multiply by 10	$10X + 10$
Subtract 1	$10X + 10 – (10 – Y)$
What is your total?	$10X + Y$

Fig 1. Card Trick 1

After the volunteer has chosen a card and placed it face down on the left, the mathemagician will choose a 9 card and place on the right of the first card face down. The mathemagician will then take the volunteer through the steps listed on the left in figure1. After being told the total by the volunteer, the mathemagician will firstly tell the audience the volunteer's hidden card and then reveal the total by turning the cards. To demonstrate instrumental understanding of the trick the mathemagician will now train the volunteer by giving the following instructions: always choose a 9; go through the steps; and when you are told the total, the number in the tens column is the hidden number. Now the volunteer is able to conduct the trick. The volunteer can continue to demonstrate the trick but if the volunteer forgets one part then the trick is lost.

This brief description reveals both the strengths and weaknesses of instrumental understanding. The trick is easy to understand but it relies upon memory and upon the mathemagician for help if the volunteer forgets.

Now examine an alternative approach. If the mathemagician helps to develop an understanding of why the trick works by explaining the steps on the right side of figure 1, then two things may happen. Firstly the trick can be adapted and secondly it is less likely to be forgotten. For example, in examining the algebraic steps it becomes obvious that the third step moves the hidden number into the 'tens' column and disguises this by supplying an extra ten. This extra is then used to produce the 9.

Pick a number from 1 - 9	• Yours is x, mine y
• Multiply by 2	• 2X
• Add 2	• 2X + 2
• Multiply by 5	• 10X + 10
• Subtract	• 10X + 10 – (10 – Y)
• What is your total?	• 10X + Y

Fig 2. Card Trick 2

Once the volunteer has this deeper understanding then variations can arise. Choosing 9 is not important as any number can be chosen. Also multiplying by ten can be broken down using the factors of ten. An example of these variations can be seen in figure 2. Notice that by step 4, the hidden number has been moved to the 'tens' column and there is an extra ten to use to get the mathemagician's number. If the mathemagician has chosen 7 then the instruction becomes 'subtract 3'. With this deeper relational understanding it is possible to vary the card trick whereby the volunteer selects both cards (see figure 3). Again the algebra shows how this third trick operates.

Pick two numbers from 1 - 9	• X and Y
• First Multiply by 2	• 2X
• Add 5	• 2X + 5
• Multiply by 5	• 10X + 25
• Add second number	• 10X + 25 + Y
• What is your total?	• 10X + Y + (25)

Fig 3. Card Trick 3

By the fourth step, the first hidden card has been moved to the 'tens' column but the extra 25 disguises this fact. By the fifth step we have the first hidden card in the 'tens' column and the second hidden card in the 'ones' column again disguised by the extra 25. Once given the total, a quick subtraction of 25 gives the two hidden cards.

Hopefully what becomes obvious from this brief discussion is the need to move the target of teaching from developing just instrumental understanding to the deeper connected relational understanding. But are the two actually separate or have we fallen into a false dichotomy similar to the early western religious philosophers and their separation of body and spirit? (see Jose Ortega y Gasset). Perhaps what is needed is a move beyond debates of 'either/or' with respect to these two types of understanding, towards 'and', recognising the complementarity of both types. This will be examined in more detail in the next section with the construction of the scale of teaching for understanding.

A Scale Of Teaching For Understanding

The targeting of teaching towards each or either type of understanding has been a concern to educational researchers. In some cases, and the writer may have been guilty of this, in emphasising the importance of relational understanding the result was that instrumental understanding was seen in a bad light. Sfard (2000) wrote:

> I decided there is a room to reconsider the idea of instrumental understanding and to ask ourselves whether our tendency to view it as a rather undesirable phenomenon is fully justified (p. 94).

Another concerns was, should teachers begin teaching for instrumental understanding then proceed to develop relational understanding? Sfard (1991) had commented that it appeared that students tended to learn mathematics initially at an instrumental level accompanied with drill and doubts, and “even professional mathematicians cannot escape this fate” (p. 32). This resonated with Skemp’s (1976) earlier comments that “even relational mathematicians often use instrumental thinking”, and it “is a point of much theoretical interest” (p. 8).

A final concern was; is it possible to start with relational understanding and then develop instrumental understanding when it is needed?

In order to attempt to answer these concerns, a model will be constructed and like all models it will be based on a number of assumptions. A common saying in mathematical modelling is ‘all models are wrong, some are useful’. A model is only useful if it is in agreement with its assumptions. So, assuming that all teaching strategies can be classified as a combination of their main focus upon one or both types of understanding, and that the struggle to assist learners to understand is akin to the struggle to assist students make sense or meaning, then it is proposed to construct a continuum that could be called: A scale of teaching for understanding or meaning.

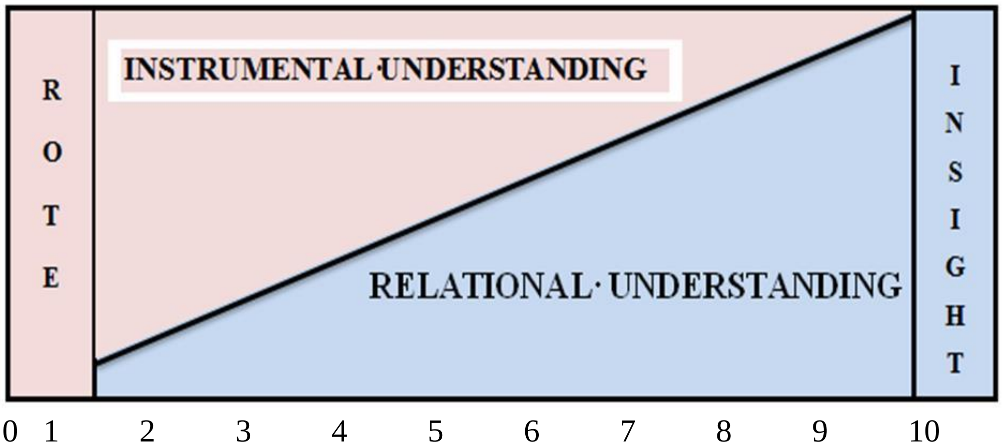

Fig 4. A Scale Of Teaching For Understanding

The left end of the scale of teaching for understanding (see figure 4) begins at the most extreme end of instrumental teaching strategies which is rote memorization. Teaching by rote memorization is defined as the memorizing of unrelated facts or procedures where there is no attempt to assist students to understand or connect what they are memorizing with what they already know and is given the score of zero on the Scale of Teaching for Understanding. Small children learn the alphabet by rote. It is much later they learn how to use this alphabet to make meaning. The term ‘rote’ is the source of considerable ‘heat’ and conflicting meanings which the following discussion will attempt to clarify.

An article with the provocative title of ‘rote is an essential feature of teaching and learning’, by two professors (Watt & McNaught, 2012) in a context of being critical of the current entry process into Australian universities, raised important issues around the process of how some mathematical facts and procedures are needed to fully understand a concept or theory and how the nature of mathematics is sequential rather than an unrelated collection. Their use of ‘rote learning’ was in the context of some ‘hard subjects’ such as mathematics requiring mastery of a sequence of knowledge that must be retained for later use in contrast with other ‘soft subjects’. They defined rote as: “Learning by repetitive confrontation with factual material …” (Watt & McNaught, 2012, p. 6). Their use of ‘rote learning’ focussed upon the outcomes rather than the process used to produce those outcomes. They did not expand upon the strategies employed in their repetitive confrontation. Nowhere in their article did they say that mathematics should be taught without meaning, merely that memory is important in the process of building more sophisticated concepts and that to build memory, a strategy of drill and practice or repetitive learning is needed.

> "There are facts, relationships, theories and concepts that must be learned, by rote since they form essential parts of students' inventories as they progress through sequences that lead to understanding" (Watt & McNaught, 2012, p. 6)

It seems clear that the left end of the scale involves the use of rote and teaching for instrumental understanding and where the deeper understanding is just starting to develop. How did you the reader learn the three essential elements of the counting process? Did you rote learn the number words and recite them in a non-negotiable order? How did you learn to tag a number with a number word using an accurate one-to-one correspondence? How did you develop an understanding that the last number in the count represents the total number of items or the cardinality of the group? When did these become meaningful for you?

Memorization can be achieved through rote chanting or repeating a phrase or formula or through a process that connects with prior knowledge. Some early childhood centres use a rote count strategy regularly throughout a day with variations of quickly/slowly; loudly/softly; steadily or in a stop/start fashion, and in isolation or with accompanying body movements. So some material has been learned using a rote strategy and is unconnected to understanding. Later it will be used to build connected understanding and it is the connected knowledge that remains longer and is ultimately of more use. Skemp (1986) provided an experiment to demonstrate the power of connected understanding by using two groups of people who learnt the names of 16 different symbols using either a rote process or a schematic process that attempted to make connections between the symbols.

> … twice as much was recalled of the schematically learnt as of the rote-learnt material when tested immediately afterwards; and in four weeks the proportion had changed to seven times as much. The schematically learnt material was not only better learnt, but better retained (pp. 39-40),

Recent developments in brain research have added further support to this work reported by Skemp.

> Students may diligently follow the teacher's instructions to memorize facts or perform a sequence of tasks repeatedly, and may even get the correct answers. But if they have not found meaning by the end of the learning episode, there is little likelihood of long-term storage (Sousa, 2008, p. 56).

There are some teaching strategies that could be identified as having a score of one. Cobb and Jackson (2011) found that many teachers 'proceduralise' problems when they launch them thus removing the problem solving objective and converting the problems to exercises in applying a procedure. Brousseau (1984) in his work on didactical contracts identified an approach where the teacher reduced a student's role by 'emptying' the task of much of its cognitive challenge. This should not be confused with the practice of 'scaffolding' which seeks to assist the student meet the challenge. This issue has serious implications for differentiated learning as what is scaffolding for one student may act as cognitive emptying for another. However, this issue is the material for a further paper.

Having established the left endpoint of the scale, it is time to consider the region between the endpoints, where the model shows a sliding division between instrumental and relational understanding. What is the justification for doing this?

There is a need to return to the recent debate over the place of rote memorization in the teaching process. Some researchers have been critical of specific Asian teaching practices denigrating them as encouraging rote memorization. Leung (2014) sought to clear up certain misconceptions around this criticism and makes a clear distinction between memorization and rote learning which is a strategy for memorization.

> Memorization may have a negative connotation for some Western educators, who see it as a sign of rote learning. But for East Asians, practice and memorization do not necessarily imply rote learning or rule out creativity. As Marton (1997) observed, in East Asia, "repetitive learning " is "continuous practice with increasing variation," and practice and repetition are considered a "route to understanding" (Hess and Azuma 1991). Biggs (1996, p. 55) pointed out that "The Chinese believe in skill development first, which typically involves repetitive, as opposed to rote learning after which there is something to be creative with. In East Asia, practice and memorization are considered legitimate (and probably effective) means for understanding and learning, and equating memorization without full understanding to rote learning may be too simplistic a view. (p. 600).

In Leung' statement, it is possible to identify a process called repetitive learning that may begin with the development of instrumental learning but gradually build relational understanding by increasing the degree of variation. So if Leung's comments are related to the scale then it may be that while East Asian teachers begin from position 1 on the scale and they gradually move their teaching to higher positions. In position 1, while it mostly involves teaching focussed upon developing instrumental understanding, the small amount of relational understanding present comes from the contribution of prior knowledge that will contribute to the meaning constructed by the learner.

In the remainder of this paper I will argue that it is this process of moving to higher levels along the scale of teaching for meaning is crucial to the development of understanding within the learner. It should be stressed that this is not an argument against memorization as students who know their multiplication tables in primary school do better at mathematics than those who do not, but usually teachers when teaching students their tables help their students develop a strong number sense and an understanding of how the number system works. While there is memorization, it takes place in a context of meaning.

It is time to consider the other extreme end of the scale. The position 9 on the scale can be observed in the ability of technology to allow a student to test a concept before learning the algebraic procedures associated with the concept. For example, an understanding of the effect 'A' has in the quadratic equation ' y=Ax2' can be developed using GeoGebra or Geometer Sketchpad dynamic geometry software, by manipulating the graph and noticing the effect upon the equation in the algebraic window.

The right endpoint of the scale of teaching for understanding with a score of 10 is the development of insight. The definition of insight is left to a famous philosopher who stated:

> By insight, then, is meant not any act of attention or advertence or memory but the supervening act of understanding (Lonergan, in Crowe & Doran, 1957, p. ix)

More recently Van Hiele wrote of the link between structure and insight from a Gestalt psychological perspective:

> "We are sure of insight when the person (or animal) you are studying comes to a conclusion on account of mental structure." In my dissertation, … (Conception and Insight), of 1957, I wrote: "Insight exists when a person acts in a new situation adequately and with intention." (Van Hiele, 1986, p. 24)

So insight is really an aim of teaching for understanding where the understanding that a student has acquired is able to be used in a novel way or upon a new task. For the Gestalt psychologists, it is the connected understanding that provides the structure for this leap of understanding. Skemp's (1987) agrees and suggests that teaching approaches are at the heart of developing relational understanding:

> By careful analysis of the mathematical structure to be acquired, we can sequence the presentation of new material in such a way that it can always be assimilated to a conceptual structure" (p. 182).

Yet having an insight is not the end of the process. A student in year 7 may come to an understanding of Pythagoras' theorem where the area of semi-circles drawn on the triangle with sides as diameters is seen as also obeying the theorem. Later in year 10, the same student may develop a further insight where Pythagoras' theorem is seen as one example (one angle equal to 90 degrees) of the more general rule known as the cosine rule (for any angles). Thus mathematics teaching that leads to the production of multiple insights in the learner is postulated as a desirable goal for the teacher.

Conclusion

This paper has sought to build a theoretical model based upon research into school mathematics teaching and learning. It was hypothesised at the beginning of this paper that mathematics teachers become expert jugglers in developing strategies that encourage students to build their mathematical understanding and develop links and connections within their knowledge, while developing their skills and positive attitudes towards their mathematical learning and knowledge. Thus in order to summarise the content of the paper, figure five is provided as a pictorial representation of what is required of a teacher whose aim if to teach for understanding in school mathematics classrooms.

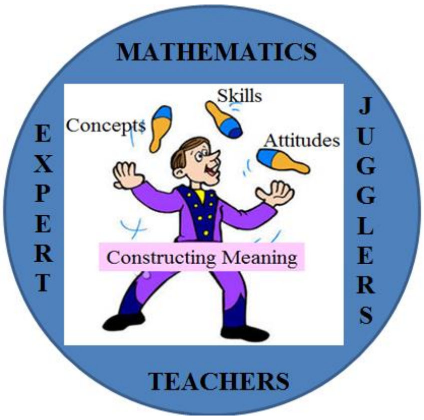

Fig 5. Juggling For Understanding And Meaning

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